THE NATURAL OSCILLATIONS OF A RECTANGULAR ELASTIC PARALLELEPIPED

(O SOBSTVENNYKH KOLEBANIIAKH PRIAMOUGOL'NOGO UPRUGOGO PARALLELEPIPEDA)

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V. V. BOLOTIN
(Moscow)

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The classical problem of the oscillations of an elastic rectangular parallelepiped has been solved only for certain particular boundary conditions [1]. There is no solution, for example, for a parallelepiped the sides of which are rigidly restrained against displacement. It will be shown that the asymptotic method previously derived for dynamic problems in the theory of plates and shells [2,3] enables us also to find a solution for certain other cases. As an example we shall consider the plane motion of a parallelepiped. The significance of Poisson's ratio ν will be illustrated, and it will be shown that for $\nu = 1/2$ the asymptotic method enables us to find all the frequencies and modes of oscillations which are related to distortion waves; for oscillations related to expansion waves the dynamic edge effect is always degenerate.

1. Let us consider the problem of the natural plane oscillations of an elastic homogeneous isotropic rectangular parallelepiped. The frequencies and modes of oscillations are defined as the characteristic values and characteristic functions of the boundary problem described by the set of equations

$$\mu\Delta u_1 + (\lambda + \mu)\left(\frac{\partial^2 u_1}{\partial x_1^2} + \frac{\partial^2 u_2}{\partial x_1 \partial x_2}\right) + \rho\omega^2 u_1 = 0$$
$$\mu\Delta u_2 + (\lambda + \mu)\left(\frac{\partial^2 u_1}{\partial x_1 \partial x_2} + \frac{\partial^2 u_2}{\partial x_2^2}\right) + \rho\omega^2 u_2 = 0 \qquad (1.1)$$

and the corresponding homogeneous boundary conditions. Here $u_1(x_1, x_2)$, $u_2(x_1, x_2)$ are the components of the displacement vector, λ and μ are Lamé's coefficients, ρ is the density, ω the frequency and Δ the two-dimensional Laplace operator. It can easily be seen that for a particular choice of constants c_1 and c_2 the set of equations (1.1) has a solution of the form

$$u_1 = c_1 \sin k_1 (x_1 - \xi_1) \cos k_2 (x_2 - \xi_2), \quad u_2 = c_2 \cos k_1 (x_1 - \xi_1) \sin k_2 (x_2 - \xi_2) \qquad (1.2)$$

where the constants k_1 and k_2 are the wave numbers and ξ_1 and ξ_2 are arbitrary phases. The characteristic equation is of the form

$$\begin{vmatrix} \Omega + k_1^2 & k_1 k_2 \\ k_2 k_1 & \Omega + k_2^2 \end{vmatrix} = 0 \qquad \left(\Omega = \frac{\mu (k_1^2 + k_2^2) - \rho\omega^2}{\lambda + \mu}\right)$$

Its roots are $\Omega_1 = 0$ and $\Omega_2 = -(k_1^2 + k_2^2)$. These roots correspond to the frequencies of the natural oscillations

$$\omega_1^2 = \frac{\mu}{\rho}(k_1^2 + k_2^2), \qquad \omega_2^2 = \frac{\lambda + 2\mu}{\rho}(k_1^2 + k_2^2) \qquad (1.3)$$

and the eigenvectors $c_1 = k_2$, $c_2 = -k_1$ and $c_1 = k_1$, $c_2 = k_2$. If we put

$$k_1 = m_1\pi / a_1, \qquad k_2 = m_2\pi / a_2 \qquad (m_1, m_2 = 1, 2, \ldots), \quad \xi_1 = \xi_2 = 0$$

solution (1.2) satisfies the boundary conditions

$$u_1 = \frac{\partial u_2}{\partial x_1} = 0 \quad \text{for } x_1 = 0 \text{ and } x_1 = a_1, \quad \frac{\partial u_1}{\partial x_2} = u_2 = 0 \quad \text{for } x_2 = 0 \text{ and } x_2 = a_2$$

where a_1 and a_2 are the lengths of the edges of the parallelepiped. This solution will be used later to give an approximate solution, with the aid of the asymptotic method developed in [2,3], for those cases of homogeneous boundary conditions which are not satisfied by (1.2) for any choice of the wave numbers k_1 and k_2 and the phases ξ_1 and ξ_2. The reasoning behind the method will not be dealt with here.

2. We shall treat the solution (1.2) as a "generating" solution which is valid within some asymptotic error only for an internal region. For the time being we can then consider the parameters k_1, k_2 (the wave numbers) and ξ_1, ξ_2 (the limiting phases) as arbitrary quantities. The problem will be solved if for a frequency ω given by (1.3) we are able to find particular solutions to the set of equations (1.1) which have the properties of the edge effect, and if their number is sufficient to satisfy all the boundary conditions on every face. Let us consider, for example, the edge $x_1 = 0$.

We shall try to find a solution which has the properties of the edge effect at this edge in the form

$$u_1 = c_1 U_1(x_1) \cos k_2 (x_2 - \xi_2), \qquad u_2 = c_2 U_2(x_1) \sin k_2 (x_2 - \xi_2) \qquad (2.1)$$

After substituting Expressions (2.1) in (1.1) we find that the functions $U_1(x_1)$ and $U_2(x_1)$ must satisfy the equations

$$\begin{aligned} &\mu c_1 (U_1'' - k_2^2 U_1) + (\lambda + \mu)(c_1 U_1'' + k_2 c_2 U_2') + \rho\omega^2 c_1 U_1 = 0 \\ &\mu c_2 (U_2'' - k_2^2 U_2) + (\lambda + \mu)(-k_2 c_1 U_1' - k_2^2 c_2 U_2) + \rho\omega^2 c_2 U_2 = 0 \end{aligned} \qquad (2.2)$$

The corresponding characteristic equation will be

$$\Delta(r, \Omega) = \begin{vmatrix} \Omega - r^2 - \dfrac{\mu}{\lambda + \mu}(r^2 + k_1^2) & -k_2 r \\ k_2 r & \Omega + k_2^2 - \dfrac{\mu}{\lambda + \mu}(r^2 + k_1^2) \end{vmatrix} = 0 \qquad (2.3)$$

Suppose that $\Omega = 0$. Then, after expanding the determinant, we find that

$$\Delta(r, 0) = \left(\frac{\mu}{\lambda + \mu}\right)^2 (r^2 + k_1^2) + \frac{\mu}{\lambda + \mu}(r^2 + k_1^2)(r^2 - k_1^2) = 0$$

The fact that the equation $\Delta(r, 0) = 0$ has roots $r_{1,2} = \pm ik_1$ is a result of the "generating" solution being taken in the form (1.2). The two other roots will be $r_{3,4} = \pm r_0$, where

$$r_0^2 = \frac{(\lambda + \mu) k_2^2 - \mu k_1^2}{\lambda + 2\mu} \qquad (2.4)$$

If $r_0^2 > 0$ one of the roots of Equation (2.3) will be negative. Then, for the edge $x_1 = 0$, we can find a solution which contains as many constants as are necessary to satisfy the boundary conditions, and which approximates to the "generating" solution (1.2) on approaching the interior of the region. Consequently, the asymptotic method is applicable if the wave numbers k_1 and k_2 satisfy the inequalities

$$(\lambda + \mu) k_1^2 - \mu k_2^2 > 0 \qquad (\lambda + \mu) k_2^2 - \mu k_1^2 > 0 \qquad (2.5)$$

Whereas the solution (1.2) satisfies the boundary conditions on a pair of opposite faces, the inequalities (2.5) do not lead to a corresponding result.

Let us consider now the solution corresponding to the frequency ω_2. Putting $\Omega = \Omega_2 = -(k_1^2 + k_2^2)$ in Equation (2.3) and rearranging, we find that

$$\Delta(r, \Omega_2) = \frac{\lambda + 2\mu}{\mu}(r^2 + k_1^2)\left[\frac{\mu}{\lambda + \mu}(r^2 + k_1^2) + k_1^2 + k_2^2\right] = 0$$

This equation has only purely imaginary roots and consequently the dynamic edge effect which corresponds to the series of frequencies ω_2 is always degenerate. Physically this means that the effect of the boundary conditions is significant for any point within the parallelepiped. This is not an unexpected result, if we consider that the frequencies ω_2

correspond to oscillations connected asymptotically with expansion waves. The problem of the oscillations of a prismatic elastic rod leads to a similar result, when account is taken of shear deformation and rotational inertia; for the second (higher) series of frequencies the dynamic edge effect is always degenerate, while for the first series of frequencies the asymptotic method gives an effective solution [4]. In this connection it is of interest to consider the following interpretation of the conditions (2.5). If we express λ and μ in terms of the elasticity modulus E and Poisson's ratio ν, conditions (2.5) become

$$k_1^2 - (1 - 2\nu) k_2^2 > 0, \qquad k_2^2 - (1 - 2\nu) k_1^2 > 0 \tag{2.6}$$

For $\nu = 1/2$ (the volume of the deformed parallelepiped remains constant during the oscillations) the dynamic edge effect is not degenerate for any values of k_1 and k_2. Conversely, when $\nu = 0$ the edge effect is always degenerate.

3. As an example let us consider the plane oscillations of a parallelepiped with the conditions that $u_1 = u_2 = 0$ on the sides $x_1 = 0$, $x_1 = a_1$, $x_2 = 0$ and $x_2 = a_2$. We first find a solution of the type (2.1) for $c_1 = k_2$, $c_2 = -k_1$, which satisfies the boundary conditions on the side $x_1 = 0$. After solving the set of equations (2.2) we find that

$$U_1 = \sin k_1 (x_1 - \xi_1) + C \exp(-r_0 x_1), \qquad U_2 = \cos k_1 (x_1 - \xi_1) - \frac{k_2^2 C}{k_1 r_0} \exp(-r_0 x_1) \tag{3.1}$$

where r_0 is a positive number given by Formula (2.4) and C is a constant. The particular solution which increases with increase in x_1 is discarded. The first terms in Expressions (3.1) correspond to the "generating" solution (1.2), the second to the dynamic edge effect.

If we now subject the solution (3.1) to the boundary conditions $U_1(0) = U_2(0) = 0$, we obtain for the limiting phase ξ_1

$$\tan k_1 \xi_1 = \frac{k_1 r_0}{k_2^2} \tag{3.2}$$

A similar relation can easily be found for the solution satisfying the conditions on the opposite side. These solutions do not coincide, but to the accuracy of a quantity of the order

$$\varepsilon \sim \exp\left(-\frac{1}{2} r_0 a_1\right) \tag{3.3}$$

they can be assumed to coincide in the limiting phases. If we equate these phases we obtain a "matching condition", which relates the wave numbers k_1 and k_2. The second solution can be found in a similar way by considering solutions which take their origins from the sides $x_2 = 0$ and

$x_2 = a_2$. From the resulting set of equations the wave numbers k_1 and k_2 can easily be determined, and the frequencies of the natural oscillations can then be found from Formula (1.3). The forms of the oscillations can be found approximately for all points in the parallelepiped,

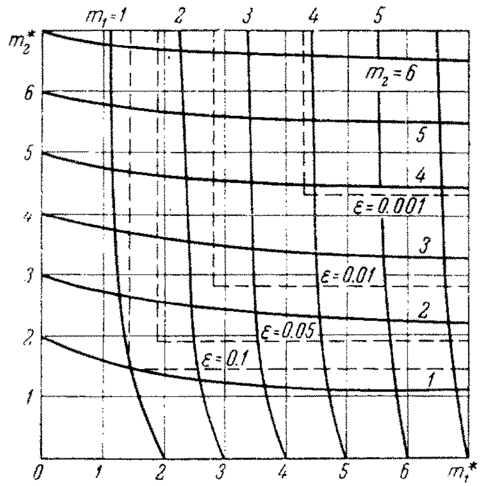

Fig. 1.

with the exception of regions immediately adjacent to its edges.

In order to find the "matching conditions", in practice it is simplest to start by considering the properties of symmetry. The forms of the oscillations in this case fall into four categories according to the nature of the symmetry. Let us consider a motion which is symmetrical along the x_1-axis. For this type of motion $U_1(1/2a_1) = U_2'(1/2a_1) = 0$. To the accuracy of a quantity ϵ given by Formula (3.3) these conditions can be replaced by the single condition $\sin k_1(1/2a_1 - \xi_1) = 0$. Thus, by making use of (3.2) we obtain the equation

$$\tan\frac{1}{2}k_1a_1 = \frac{k_1r_0}{k_2^2} \tag{3.4}$$

For motions which are antisymmetric along the x_1-axis we have the conditions $U_1'(1/2a_1) = U_2(1/2a_1) = 0$. These conditions will be satisfied to the accuracy of ϵ if we set $\cos k_1(1/2a_1 - \xi_1) = 0$. Thus

$$\cot\frac{1}{2}k_1a_1 = -\frac{k_1r_0}{k_2^2} \tag{3.5}$$

Equations (3.4) and (3.5) can be combined into the single equation

$$k_1a_1 = 2\tan^{-1}\frac{k_1}{k_2^2}\left[\frac{k_2^2-(1-2\nu)k_1^2}{2(1-\nu)}\right]^{1/2} + m_1\pi \quad (m_1 = 1, 2, \ldots) \tag{3.6}$$

which is derived by making use of Formula (2.4). For the function $\tan^{-1}$ its principal values are taken. The second equation is derived from (3.6) by cyclic permutation of indices:

$$k_2a_2 = 2\tan^{-1}\frac{k_2}{k_1^2}\left[\frac{k_1^2-(1-2\nu)k_2^2}{2(1-\nu)}\right]^{1/2} + m_2\pi \qquad (m_2 = 1, 2, \ldots) \tag{3.7}$$

It can be seen from Equations (3.6) and (3.7) that the wave numbers and frequencies in regions where there is no degeneration of the edge effect have the same asymptotic behavior as in the problem of oscillations of a membrane [5]:

$$k_1a_1/\pi = m_1 + O(1), \qquad k_2a_2/\pi = m_2 + O(1) \qquad (m_1, m_2 = 1, 2, \ldots)$$

In contrast to the asymptotic methods of Courant-Weyl [5], Equations (3.6) and (3.7) enable us to find the wave numbers to the accuracy of ϵ. We note that on the boundary of the region of degeneration $k_2^2 - (1-2\nu)k_1^2 = 0$ Equation (3.6) has an exact solution $k_1a_1/\pi = m_1$, and on the other boundary Equation (3.7) has the solution $k_2a_2/\pi = m_2$. With k_1 fixed, as $k_2 \to \infty$, $k_1a_1/\pi \to m_1$; similarly with k_2 fixed, as $k_1 \to \infty$, $k_2a_2/\pi \to m_2$. If $m_1 = m_2 = m$, $a_1 = a_2 = a$ the set of equations (3.6) and (3.7) has a solution $k_1 = k_2 = k$, where

$$ka = 2\tan^{-1}\left(\frac{\nu}{1-\nu}\right)^{1/2} + m\pi$$
$$(m = 1, 2, \ldots)$$

In particular, if $\nu = 1/2$, then

$$ka/\pi = m + 1/2.$$

In this way it is not difficult to imagine a diagram of the distribution of the roots of (3.6) and (3.7) in the plane $m_1^* = k_1a_1/\pi$, $m_2^* = k_2a_2/\pi$. Such a diagram for the case of $\nu = 1/2$ is shown in Fig. 1. The roots of the set of equations are defined as the coordinates of the points of intersection of the thick continuous lines (their values can be found more precisely by iteration). The thin continuous lines correspond to the case of boundary conditions for which the generating solution (1.2) is the exact solution. The lines along which the *a priori* error given by Formula (3.8) has the values (0.1, 0.05, 0.01 and 0.001) are shown dotted. Figure 2 shows a similar diagram for the case of $\nu = 0.35$, $a_1 = a_2$. The region of degeneration, within which the asymptotic method is inapplicable, is shown hatched. The error of this method rapidly diminishes with increase in distance from the edge of the region of degeneration.

4. In the foregoing we have considered only the case of plane deformation. In order to apply the results to the case of a state of plane stress (a thin plate performing oscillations in its own plane), we simply replace λ by $\lambda^* = \nu E/(1-\nu^2)$ in Equations (1.1) and in subsequent formulas containing λ and μ. For example, instead of the conditions (2.6) for the applicability of the method, we have the conditions $k_1^2(1+\nu) - k_2^2(1-\nu) > 0$, $k_2^2(1+\nu) - k_1^2(1-\nu) > 0$, etc.

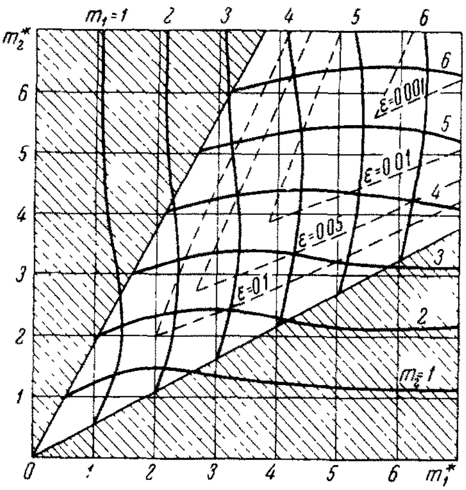

Fig. 2.

In the case of three-dimensional oscillations the "generating" solution corresponds to three series of frequencies: two series related to distortion waves, and one related to expansion waves. For the latter the dynamic edge effect is always degenerate. For the first two series, instead of conditions (2.6) we have the condition $k_1^2 + k_2^2 - (1-2\nu)k_3^2 > 0$ and a further two conditions derived by cyclic permutation of indices. For $\nu = 1/2$ the edge effect is not degenerate for any combination of k_1, k_2 and k_3.

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